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SPECTRAL REGULARIZATION OF INTEGRAL EQUATIONS IN THE THEORY OF ELASTICITY*

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It is well-known that singular integral operators in spatial problems in the theory of elasticity have a spectrum with three significant points. Regularization is carried out by transforming the original equation so as to combine these points into one. An explicit description of regularizers and regularized equations is given.

The proof of the applicability of Fredholm alternatives to singular integral equations in the theory of elasticity /1, 2/ rests on the proof that it is in principle possible to regularize these equations.

For an explicit construction of regularizers and regularized equations we study the spectra of appropriate singular operators.

For the first internal problem (1^+) and the first external problem (1^-) of the theory of elasticity, the integral equations have the form:

$$(\mathbf{a}\mathbf{I} - \mathbf{A})\,\varphi = \mathbf{V} \tag{1}$$

where I is the identity operator, **A** is a singular operator, **V** is the transformation vector for the point **y** of the surface S of the elastic body, $\mathbf{a} = \pm 1$ (where $\mathbf{a} = +1$ for problem 1^+) and φ is the density vector of a potential which is used to calculate the transformation of the point of the elastic body /3/.

The action of the operator **A** is given by the equation

$$A\varphi = \int\limits_S \varphi(x)\,\Phi(x, y)\,dS_x$$

$$\Phi(x, y) = \frac{1}{4\pi(1-\nu)|r|^3}\left[(1-2\nu)(n_x r - r n_x - n_x \cdot r I) - 3\frac{n_x r}{|r|^3}\,rr\right]$$

$\Phi(x, y)$ is a tensor of rank two, ν is Poisson's ratio, $r = x - y$, where x and y are radius vectors of points on the surface S and n_x is the external normal to the surface S at the point x. In the text, we use the notation of direct tensor calculus /3/, with the exception that tensors and vectors are not identified by special notation but are determined by the context.

The kernel $\Phi(x, y)$ contains a singularity of type $1/|r|^2$, thus, the integral operator A is singular. Regularization of Eq.(1) requires the existence of a singular operator B which satisfies the equation

$$B(aI - A)\,\varphi = (I - T)\,\varphi = -BV \tag{2}$$

where T is a fully continuous operator.

The operator B is called the regularizer of Eq.(1). If Eqs.(1) and (2) are equivalent, then B is an equivalent regularizer.

For an explicit construction of the regularizer B and a regularized Eq.(2), according to the scheme described in this paper, we initially need to study the points at which the spectrum of the operator A is continuous.

Theorem 1. The spectrum of the integral operator A from the theory of elasticity contains three points at which it is continuous.

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm b;\; b = \frac{1-2\nu}{2(1-\nu)} \tag{3}$$

The proof is based on a consideration of the symbolic matrix of the operator $\lambda I - A$. According to Mikhlin /1/, the symbolic matrix of this operator for an arbitrary angle θ may

Prikl. Matem. Mekhan., 55, 2, 348-351, 1991 be written in the form (*Mal'tsev L.E. and Kurilenko E.Yu., Spectral representation of a singular integral operator in the theory of elasticity, Unpublished Paper, VINITI 5.02.86, 841-B86, 1985.)

$$\begin{Vmatrix} \lambda & 0 & \mathbf{bi}\cos\theta \\ 0 & \lambda & \mathbf{bi}\sin\theta \\ -\mathbf{bi}\cos\theta & -\mathbf{bi}\sin\theta & \lambda \end{Vmatrix}$$

Equating the corresponding symbolic determinant to zero we obtain the equation $\lambda(\lambda^2 - b^2) = 0$, whence we obtain the three values of λ.

For these values of λ, it is not possible to regularize the equation $(\lambda I - A)\varphi = f$ (Theorem 1.40 of /1/); moreover, these numbers belong to the spectrum of the operator A (Theorem 3.24 /1/). Consequently, we have identified three singular points. These can only be accumulation points of the spectrum and cannot be points of infinite multiplicity. The finite multiplicity of points of the spectrum of the operator A is proved in /2/.

The existence of not one but three accumulation points of the spectrum of the singular integral operator is a differentiating feature compared with arbitrary fully-continuous operators. This difference enables us to regularize Eq.(1) and to find a transformation of it in which the three points of continuity of the spectrum are transformed into a singular point which coincides with zero.

Theorem 2. The operator

$$B = \frac{1}{\mathbf{a}} I + \frac{1}{1-\mathbf{b}^2} A + \frac{1}{\mathbf{a}(1-\mathbf{b}^2)} a^2 \tag{4}$$

is a regularizer of Eq.(1); the regularized equation has the form

$$\varphi - \frac{1}{\mathbf{a}(1-\mathbf{b}^2)} A(A^2 - b^2 I)\varphi = -BV \tag{5}$$

To prove this, we apply a theorem on spectral transformations /4/, which holds for linear bounded operators in Banach spaces. Suppose that $\mathbf{P}_n(x)$ is a polynomial of degree n and that A is an operator, the spectrum of which is determined by the set $\{\lambda\}$. Then the spectrum of the operator $\mathbf{P}_n(A)$ is determined by the set $\{P_n(\lambda)\}$.

Thus, for Eq.(2), for the regularizer B we should take a polynomial of degree two with arbitrary coefficients c, d and e.

$$B = \mathbf{c}I + \mathbf{d}A + \mathbf{e}A^2$$

Substituting this operator into (2) gives

$$\mathbf{B}(\mathbf{a}\mathbf{I} - \mathbf{A})\varphi = (\mathbf{ac}\mathbf{I} - \mathbf{T})\varphi$$
$$\mathbf{T} = (\mathbf{c} - \mathbf{ad})\mathbf{A} + (\mathbf{d} - \mathbf{ae})\mathbf{A}^2 + \mathbf{e}\mathbf{A}^3$$

Comparison with Eq.(2) shows that in order to carry out the regularization we require that $\mathbf{ac} = 1$ and the operator $\mathbf{T}$ must have a single spectral accumulation point equal to zero. According to the theorem on spectral transformations, for this we require that the polynomial of degree three

$$\mathbf{P}_3(\lambda) = (\mathbf{c} - \mathbf{ad})\lambda + (\mathbf{d} - \mathbf{ae})\lambda^2 + \mathbf{e}\lambda^3$$

should map to zero at the three points $\lambda_1 = 0$ and $\lambda_{2,3} = \pm b$.

This condition will be satisfied if we set

$$c = 1/a,\ d = 1/(1 - b^2),\ e = d/a$$

This choice of coefficients determines the regularizer (4).

Application of the regularizer (4) to Eq.(1) leads to regularization of Eq.(5).

Integral equations of the second internal problem (2^+) and second external problem (2^-) in the theory of elasticity are described in terms of the operator A^*, the conjugate of A

$$(hI - A^*)\varphi = -F \tag{6}$$

where $h = \pm 1$ (and $h = +1$ for 2^- problems), and F is the pressure vector at points of the boundary S of the elastic body.

The operator A^* acts according to the rule

$$A^*\varphi = \int_S \Phi(x, y)\cdot\varphi(x)\, dS_x$$

Theorem 3. For elastic bodies bounded by a Lyapunov surface S, the operators A and A^* are distinguished by full continuity.

The proof is given in /5/.

Theorem 4. The continuous spectra of the operators A and A^* are identical.

The proof follows from the fact that operator symbols differing by a fully continuous operator are identical /1/.

Theorem 5. The operator

$$B^* = \frac{1}{h} I + \frac{1}{1-b^2} A^* + \frac{1}{h(1-b^2)} A^{*2} \tag{7}$$

is a regular of Eq.(7). The regularized equation has the form

$$\varphi - \frac{1}{h(1-b^2)} A^* (A^{*2} - b^2 I)\varphi = -B^* F \tag{8}$$

The proof is analogous to that of Theorem 2. The theorem of /4/ is used.

Theorem 6. In regularizers (4) and (7) the operators are interchangeable.

To prove this, we replace the operator A in (5) by the operator A^* and apply Theorem 3. The equation obtained differs from Eq.(5) in a fully continuous term. As is well-known /1/, if a fully continuous operator is added to another operator, the property that the second operator is a regularizer is preserved. This completes the proof of Theorem 6.

Theorem 7. The regularizers (4) and (7) are equivalent regularizers.

To prove this, it is sufficient to show that the regularizers B and B^* do not have zeros (except the trivial zero) /1/.

Suppose, for example, that $B\varphi = 0$. This means that the number $\beta = 0$ belongs to the spectrum of the operator B. Since the operator B is a polynomial of degree two in the operator A, from the theorem on the spectral transformations, we obtain

$$\beta = \frac{1}{a} + \frac{1}{1-b^2}\lambda + \frac{1}{a(1-b^2)}\lambda^2$$

where $\{\lambda\}$ and $\{\beta\}$ are the spectra of the operators A and B. Setting $\beta = 0$ and solving the quadratic equation, we obtain $\lambda = (-a \pm \sqrt{4b^2-3})/2$.

For real Poisson's ratios $0 \leqslant \nu \leqslant 1/2$, the number λ is complex, which contradicts the fact that the spectrum of the operator A is known to be real /2/. Thus $\beta \neq 0$ and the equation $B\varphi = 0$ does not have non-trivial solutions. The equivalence of the regularizer B^* is proved analogously.

It follows from the above that the difficulties in constructing a regularizer are associated with a detailed study of the spectrum. In addition to the strengthening of regularization techniques an explicit description of regularized equations is also of interest. For example, it was used in /6/ as a basis for a method of mechanical quadrature applied to integral equations in the theory of elasticity.

We note that the regularizer (4) and the regularized Eq.(5) were previously constructed /7/ without using spectral information.

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